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LARGE ELASTIC DEFORMATIONS OF ISOTROPIC MATERIALS. II. SOME UNIQUENESS THEOREMS FOR PURE, HOMOGENEOUS DEFORMATION

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The equilibrium of a cube of incompressible, neo-Hookean material, under the action of three pairs of equal and oppositely directed forces f_1, f_2, f_3, applied normally to, and uniformly distributed over, pairs of parallel faces of the cube, is studied. It is assumed that the only possible equilibrium states are states of pure, homogeneous deformation.

It is found that

(1) when the stress components in the deformed cube are specified, the corresponding equilibrium state is uniquely determined (this is shown in § 6 of Part I).

(2) When the three pairs of equal and oppositely directed forces f_1, f_2 and f_3 are specified,

(*a*) the corresponding equilibrium state is uniquely determined, provided that one or more of the forces f_1, f_2 and f_3 is negative, i.e. is a compressional force, or, if they are all positive, provided that

$$f_1 f_2 f_3 < (\tfrac{1}{3}E)^3,$$

where $\frac{1}{3}E$ is the constant of proportionality between the stress and strain components (analogous to the rigidity modulus of the classical theory of small elastic deformations of isotropic materials).

(*b*) If f_1, f_2 and f_3 are all positive and

$$f_1 f_2 f_3 > (\tfrac{1}{3}E)^3,$$

then the equilibrium state is not necessarily uniquely determined. The number of equilibrium states which exist depends on the values of f_1, f_2, f_3 and $\frac{1}{3}E$. The actual state of deformation which is obtained depends in general on the order in which the forces are applied.

In the cases (1) and (2*a*), it is shown that the unique equilibrium state is one of stable equilibrium.

In case (2*b*), it is shown that the possible equilibrium states are of eight different types. Two of these are mutually exclusive. Of these eight types of equilibrium state, four are inherently unstable. Of the other four types, one is inherently stable. The three remaining types of equilibrium state are not necessarily unique; i.e. more than one equilibrium state of each type may correspond to specified values of f_1, f_2 and f_3. However, if one or more equilibrium states of a particular type exists, then at least one of them is stable.

If $f_1 = f_2 = f_3 = f$, then there is a unique state of stable equilibrium, provided that

$$f < \tfrac{1}{3}E.$$

If

$$\tfrac{1}{3}E < f < (\tfrac{1}{4})^{\frac{1}{3}} E,$$

then there are three states of stable equilibrium, none of which is identical with the undeformed state of the cube. If

$$(\tfrac{1}{4})^{\frac{1}{3}} E < f < \tfrac{2}{3}E,$$

there are four states of stable equilibrium, one of which is identical with the undeformed state of the cube. If

$$f > \tfrac{2}{3}E,$$

then there is only one state of stable equilibrium, and this is identical with the undeformed state of the cube.

1. Introduction

The most general deformation of any highly elastic material may be considered to consist of a pure rotation followed by a pure, homogeneous strain. The amounts of the pure rotation and of the pure, homogeneous strain may, of course, vary from point to point of the material.

The components $(t_{xx}, t_{yy}, t_{zz}, t_{yz}, t_{zx}, t_{xy})$ of the stress at any point, which has co-ordinates (x, y, z) in the undeformed state, referred to a rectangular, Cartesian system of co-ordinates, must then be related to the components $(\epsilon'_{xx}, \epsilon'_{yy}, \epsilon'_{zz}, \epsilon'_{yz}, \epsilon'_{zx}, \epsilon'_{xy})$ of this pure, homogeneous portion of the total strain at that point. If u, v and w are the components, parallel to the axes x, y and z, of the displacement undergone by the point considered, in the deformation, then $(\epsilon'_{xx}, \epsilon'_{yy}, \ldots)$ are given by equations (I, 3·9) of the foregoing paper.

In the foregoing paper, the concept of an incompressible, neo-Hookean material has been formulated. Such a material is considered to be isotropic in its undeformed state and to obey the stress-strain relations (I, 3·10). It is readily seen that if the strain components $(\epsilon'_{xx}, \epsilon'_{yy}, \ldots)$ are specified, the hydrostatic pressure p is indeterminate. This results from the fact that the material is incompressible. On the other hand, it is shown (I, § 5) that if the stress components $(t_{xx}, t_{yy}, \ldots)$ are specified, the hydrostatic pressure p can be determined in terms of them by means of a cubic equation, which may yield one or three real values for p, depending on the magnitudes of the stress components and of the physical constant E.

It is shown, however, (I, § 6) that only one of these values of p—the least—leads to real values of the strain components $(\epsilon'_{xx}, \epsilon'_{yy}, \ldots)$. It is thus apparent that if a cube of incompressible, neo-Hookean material is subjected to a pure, homogeneous deformation and the stress components in the deformed state are specified, the state of strain of the deformed cube is uniquely determined. This fact can be used to prove the further theorem that if, in an element of an incompressible, neo-Hookean material, which has the form of a cube in its undeformed state, the total forces, acting normally over each pair of parallel faces, are specified, then the components of strain are uniquely determined provided that one or more of the forces acting is negative, or, if they are all positive, their geometric mean is less than $\frac{1}{3}E$ (§ 2). If all the forces are positive and their geometric mean is greater than $\frac{1}{3}E$, then the possibility of more than one equilibrium state exists. These equilibrium states are of eight different types. The relations that must be satisfied by these three applied forces, for these different types of equilibrium state to exist, are examined (§§ 3, 4). The conditions for the stability of the equilibrium states in the various cases discussed are considered (§§ 5 to 8). The case when the three applied forces are equal is then discussed (§ 9). Finally, the manner in which the order of application of the forces influences the equilibrium condition reached is discussed (§ 10).

2. Uniqueness theorem for specified forces

Let us consider a volume of the incompressible, neo-Hookean material which, in the unstrained state, had the form of a cube of unit edge. Suppose that this cube is subjected to a pure, homogeneous strain, under the action of three pairs of equal and oppositely-directed forces, f_1, f_2 and f_3, mutually at right angles. In the strained state, each of these forces is considered to be uniformly distributed over the plane face of the cuboid on which it acts and is, of course, directed normally to that face. In this section, we shall prove that if the values of f_1, f_2 and f_3 are specified, the physically possible pure, homogeneous strain produced is uniquely determined, provided that one, or more, of f_1, f_2 and f_3 are negative, or they are all positive and

$$f_1 f_2 f_3 < \tfrac{1}{27}E^3.$$

Taking as axes of reference a rectangular, Cartesian, co-ordinate system x, y, z, whose axes are parallel to the edges of the cube, the relation between the stress components and the lengths, λ_1, λ_2 and λ_3, of the edges of the deformed body is given by equations (I, 3·5), the hydrostatic pressure p being given by equation (I, 6·1). In these equations X, Y and Z must now be replaced by x, y and z respectively.

The stress components t_{xx}, t_{yy} and t_{zz} are determined in terms of the forces f_1, f_2 and f_3, by the relations

$$t_{xx} = f_1/\lambda_2\lambda_3 = \lambda_1 f_1, \quad t_{yy} = f_2/\lambda_3\lambda_1 = \lambda_2 f_2 \quad \text{and} \quad t_{zz} = f_3/\lambda_1\lambda_2 = \lambda_3 f_3. \tag{2·1}$$

Combining equations (2·1) and (I, 3·5), we have

$$\lambda_1 f_1 = \tfrac{1}{3}E\lambda_1^2 + p, \quad \lambda_2 f_2 = \tfrac{1}{3}E\lambda_2^2 + p \quad \text{and} \quad \lambda_3 f_3 = \tfrac{1}{3}E\lambda_3^2 + p. \tag{2·2}$$

In principle, p could be determined by using the incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$ to eliminate λ_1, λ_2 and λ_3 in these equations. However, this would be extremely cumbersome to carry out and, in fact, the uniqueness of the physically possible values of λ_1, λ_2 and λ_3, for given values of f_1, f_2 and f_3, can be proved without resorting to this analysis.

Suppose that there are two possible sets of values of λ_1, λ_2 and λ_3. Let us denote these λ_1', λ_2', λ_3' and λ_1'', λ_2'', λ_3''. Let the corresponding values of the stress components be t'_{xx}, t'_{yy}, t'_{zz} and t''_{xx}, t''_{yy}, t''_{zz} respectively and of the hydrostatic pressure p be p' and p'' respectively. Then, from equations (2·1) and (2·2),

$$t'_{xx} = \lambda_1' f_1 = \tfrac{1}{3}E\lambda_1'^2 + p', \quad t'_{yy} = \lambda_2' f_2 = \tfrac{1}{3}E\lambda_2'^2 + p' \quad \text{and} \quad t'_{zz} = \lambda_3' f_3 = \tfrac{1}{3}E\lambda_3'^2 + p'. \tag{2·3}$$

Also

$$t''_{xx} = \lambda_1'' f_1 = \tfrac{1}{3}E\lambda_1''^2 + p'', \quad t''_{yy} = \lambda_2'' f_2 = \tfrac{1}{3}E\lambda_2''^2 + p'' \quad \text{and} \quad t''_{zz} = \lambda_3'' f_3 = \tfrac{1}{3}E\lambda_3''^2 + p''. \tag{2·4}$$

The incompressibility of the material implies that

$$\lambda_1'\lambda_2'\lambda_3' = \lambda_1''\lambda_2''\lambda_3'' = 1.$$

Consequently,

$$t'_{xx}t'_{yy}t'_{zz} = t''_{xx}t''_{yy}t''_{zz}. \tag{2·5}$$

p' is a solution of the equation

$$(p - t'_{xx})(p - t'_{yy})(p - t'_{zz}) + \tfrac{1}{27}E^3 = 0, \tag{2·6}$$

and p'' is a solution of the equation

$$(p - t''_{xx})(p - t''_{yy})(p - t''_{zz}) + \tfrac{1}{27}E^3 = 0. \tag{2·7}$$

It has already been seen (in I, § 6) that, if there is more than one real value of p which satisfies either of these equations, then only the least of these leads to real values for the dimensions of the strained body.

The solutions of equation (2·6) are the values of p for which the cubic

$$P = (p - t'_{xx})(p - t'_{yy})(p - t'_{zz}) \tag{2·8}$$

and the straight line

$$P = -\tfrac{1}{27}E^3 \tag{2·9}$$

intersect.

Again, the solutions of equation (2·7) are the values of p for which the cubic

$$P = (p - t''_{xx})(p - t''_{yy})(p - t''_{zz}) \tag{2·10}$$

and the straight line (2·9) intersect.

The two cubics (2·8) and (2·10) intersect where

$$(p-t'_{xx})(p-t'_{yy})(p-t'_{zz}) = (p-t''_{xx})(p-t''_{yy})(p-t''_{zz}). \quad (2\cdot11)$$

In view of the relation (2·5), (2·11) becomes

$$p[p\{(t'_{xx}+t'_{yy}+t'_{zz})-(t''_{xx}+t''_{yy}+t''_{zz})\} \\ -\{(t'_{yy}t'_{zz}+t'_{zz}t'_{xx}+t'_{xx}t'_{yy})-(t''_{yy}t''_{zz}+t''_{zz}t''_{xx}+t''_{xx}t''_{yy})\}]=0.$$

Thus, the two cubics intersect at only two real, finite points

$$p=0 \quad \text{and} \quad p=\frac{(t'_{yy}t'_{zz}+t'_{zz}t'_{xx}+t'_{xx}t'_{yy})-(t''_{yy}t''_{zz}+t''_{zz}t''_{xx}+t''_{xx}t''_{yy})}{(t'_{xx}+t'_{yy}+t'_{zz})-(t''_{xx}+t''_{yy}+t''_{zz})}.$$

Therefore, if the two cubics (2·8) and (2·10) are plotted on the same graph, they will intersect on the P-axis. The cubic (2·8) cuts the p-axis where $p = t'_{xx}$, t'_{yy} and t'_{zz} and the cubic (2·10) cuts it where $p = t''_{xx}$, t''_{yy} and t''_{zz}. We note that the λ's in equations (2·3) and (2·4) are essentially positive, so that t'_{xx}, t'_{yy} and t'_{zz} have the same signs as t''_{xx}, t''_{yy} and t''_{zz} respectively and each of these sets of stress components has the same signs as f_1, f_2 and f_3 respectively.

Plotting equations (2·8) and (2·10) on the same graph we have curves I and II of figure 1.

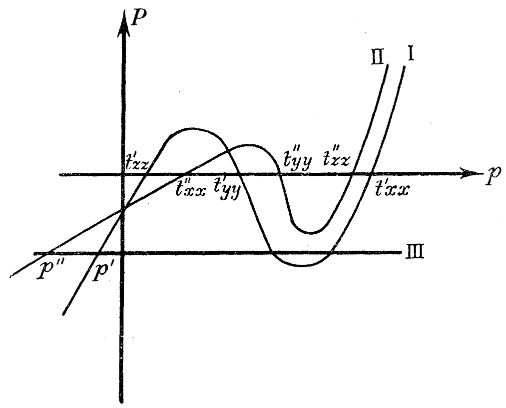

FIGURE 1

In order to be explicit, in figure 1, f_1, f_2 and f_3—and hence all the stress components—have been taken positive, and, we have assumed

$$t'_{xx} > t'_{yy} > t'_{zz} \quad \text{and} \quad t''_{zz} > t''_{yy} > t''_{xx}.$$

The argument is unaltered if any other order for the stress components is taken and if one, or more, of the forces f_1, f_2 and f_3 is negative. In figure 1, the straight line (2·9) is drawn as curve III and the values of p' and p'', which must be taken in equations (2·3) and (2·4) respectively, are given by the lowest values of p for which it intersects curves I and II respectively. It can be readily seen that p' and p'' must have the same sign.

This will be negative, if one or more of the forces f_1, f_2 and f_3 is negative or if they are all positive and the straight line III cuts the P-axis below its intersection with the cubics I and II, i.e. if

$$-\tfrac{1}{27}E^3 < -f_1 f_2 f_3. \quad (2\cdot12)$$

Again, p' and p'' will both be positive if f_1, f_2 and f_3 are all positive and if the straight line III cuts the P-axis above its intersection with I and II, i.e. if

$$-\tfrac{1}{27}E^3 > -f_1 f_2 f_3. \quad (2\cdot13)$$

Let us first consider the case when p' and p'' are both negative.

From equations (2·3),

$$2E\lambda_i' = 3[f_i \pm (f_i^2 - \tfrac{4}{3}Ep')^{\frac{1}{2}}] \quad (i = 1, 2, 3). \tag{2·14}$$

Again, from equations (2·4),

$$2E\lambda_i'' = 3[f_i \pm (f_i^2 - \tfrac{4}{3}Ep'')^{\frac{1}{2}}] \quad (i = 1, 2, 3). \tag{2·15}$$

Since the λ's are essentially positive, to be physically possible, and since p' and p'' are negative, only the positive square roots in (2·14) and (2·15) are allowable. So, equations (2·14) become

$$2E\lambda_i' = 3[f_i + (f_i^2 - \tfrac{4}{3}Ep')^{\frac{1}{2}}] \quad (i = 1, 2, 3), \tag{2·16}$$

and equations (2·15) become

$$2E\lambda_i'' = 3[f_i + (f_i^2 - \tfrac{4}{3}Ep'')^{\frac{1}{2}}] \quad (i = 1, 2, 3). \tag{2·17}$$

Now suppose $p' > p''$. Then, $\lambda_1' < \lambda_1''$, $\lambda_2 < \lambda_2''$ and $\lambda_3' < \lambda_3''$.

This means that $\lambda_1'\lambda_2'\lambda_3' < \lambda_1''\lambda_2''\lambda_3''$. But, since the material is incompressible,

$$\lambda_1'\lambda_2'\lambda_3' = \lambda_1''\lambda_2''\lambda_3'' = 1.$$

Consequently, $$p' \leqslant p''.$$

By a similar *reductio ad absurdum* argument, it can be shown that $p' \geqslant p''$. Therefore $p' = p''$.

The pure, homogeneous strain is thus uniquely determined, if the three forces f_1, f_2 and f_3 are specified and are such that one, or more, of them is negative, or they are all positive and the condition (2·12)

$$f_1 f_2 f_3 < \tfrac{1}{27}E^3$$

is valid.

In particular, we notice that when one, or more, of the forces f_1, f_2 and f_3 are zero, the condition (2·12) is satisfied and the strain is uniquely determined if the applied forces are specified.

3. The case of multiple solutions

Now, let us consider the case when f_1, f_2 and f_3 are all positive and

$$f_1 f_2 f_3 > \tfrac{1}{27}E^3$$

is satisfied, i.e. when p' and p'' are positive.

From equations (2·14), we see that a positive value of p' will lead to real values of λ_1', λ_2' and λ_3', only if p' is less than the least of the quantities $3f_1^2/4E$, $3f_2^2/4E$ and $3f_3^2/4E$. Similarly, from equations (2·15), a positive value of p'' will lead to real values of λ_1'', λ_2'' and λ_3'', only if p'' is less than the least of the quantities $3f_1^2/4E$, $3f_2^2/4E$ and $3f_3^2/4E$. If both of these conditions are satisfied, then either the positive or negative alternatives for the square roots in equations (2·14) and (2·15) will lead to real, positive values for the λ's.

It can readily be seen by an argument similar to that of the previous section, that there can only be one solution to the problem, in which all the square roots are taken positive, i.e. of the type

$$2E\lambda_i' = 3[f_i + (f_i^2 - \tfrac{4}{3}Ep')^{\frac{1}{2}}] \quad (i = 1, 2, 3). \tag{3·1}$$

Also, it can be seen, by a similar argument, that there is only one solution, in which all the square roots are taken negative, i.e. of the type

$$2E\lambda_i'' = 3[f_i - (f_i^2 - \tfrac{4}{3}Ep'')^{\frac{1}{2}}] \quad (i = 1, 2, 3). \tag{3·2}$$

Further, the existence of a solution of the type (3·1) excludes a solution of the type (3·2), since if two such solutions exist

$$\lambda_1'' < \lambda_1', \quad \lambda_2'' < \lambda_2' \quad \text{and} \quad \lambda_3'' < \lambda_3',$$

giving

$$\lambda_1''\lambda_2''\lambda_3'' < \lambda_1'\lambda_2'\lambda_3',$$

which contradicts the incompressibility condition.

Conversely, by a similar argument, the existence of a solution of the type (3·2) excludes a solution of the type (3·1).

The conditions, under which solutions of the types (3·1) and (3·2) are possible, may be readily found. Let us consider that $f_1 > f_2 > f_3$.

We may plot the equations

$$\lambda_1 f_1 = \tfrac{1}{3}E\lambda_1^2 + p, \tag{3·3}$$

$$\lambda_2 f_2 = \tfrac{1}{3}E\lambda_2^2 + p \tag{3·4}$$

and

$$\lambda_3 f_3 = \tfrac{1}{3}E\lambda_3^2 + p \tag{3·5}$$

on a single diagram (figure 2) treating them as relationships between p and λ_1, λ_2 and λ_3 respectively. The axes of λ_1, λ_2 and λ_3 are the common axis λ. Curves I, II and III represent equations (3·3), (3·4) and (3·5) respectively. The points A, B and C, at which I, II and III respectively cut the λ-axis, occur when $\lambda_1 = 3f_1/E$, $\lambda_2 = 3f_2/E$ and $\lambda_3 = 3f_3/E$. The vertices of the parabolae I, II and III occur when $\lambda = 3f_1/2E$, $p = 3f_1^2/4E$, when $\lambda = 3f_2/2E$, $p = 3f_2^2/4E$ and when $\lambda = 3f_3/2E$, $p = 3f_3^2/4E$.

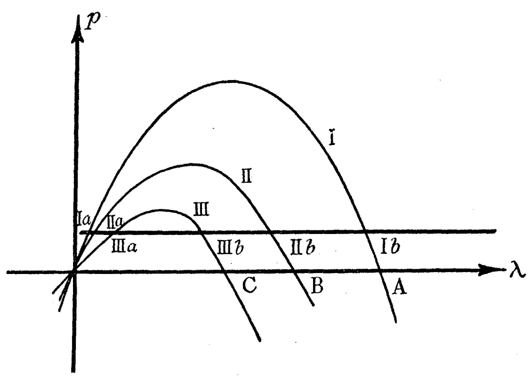

FIGURE 2

Provided $0 < p' < 3f_3^2/4E$, the straight line $p = p'$ cuts each of the curves I, II and III in two points. Let us denote the lower of these two points by Ia, IIa and IIIa respectively and the upper by Ib, IIb and IIIb respectively, giving two values of λ_1, two values of λ_2 and two values of λ_3. Only for certain values of p' can values of λ_1, λ_2 and λ_3, which make $\lambda_1\lambda_2\lambda_3 = 1$, be selected from these alternatives.

Since, as p' increases, Ib, IIb and IIIb all decrease, the maximum value of $\lambda_1\lambda_2\lambda_3$ is given by $p' = 0$ and the minimum value by $p' = 3f_3^2/4E$, for which $p = p'$ is tangential to curve III at its vertex. Provided 1 lies between these two extreme values of $\lambda_1\lambda_2\lambda_3$, a solution of the type Ib, IIb, IIIb exists. This condition is

$$\frac{27f_1f_2f_3}{E^3} > 1 > \left(\frac{3}{2E}\right)^3 f_3[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1+(f_1^2-f_3^2)^{\frac{1}{2}}]. \tag{3·6}$$

Again, as p' increases, Ia, IIa and IIIa all increase. The minimum value of $\lambda_1\lambda_2\lambda_3$ is zero and its maximum is

$$\left(\frac{3}{2E}\right)^3 f_3[f_2-(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}].$$

Thus provided
$$1 < \left(\frac{3}{2E}\right)^3 f_3[f_2-(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}], \tag{3·7}$$

a solution of the type Ia, IIa, IIIa exists.

Combining conditions (3·6) and (3·7), it is clear that if

$$\left(\frac{3}{2E}\right)^3 f_3[f_2-(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}] < 1 < \left(\frac{3}{2E}\right)^3 f_3[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1+(f_1^2-f_3^2)^{\frac{1}{2}}], \tag{3·8}$$

then neither a solution of the type Ia, IIa, IIIa, nor of the type Ib, IIb, IIIb exists.

Apart from the two types of solution

(i) Ia, IIa, IIIa

and (ii) Ib, IIb, IIIb,

which have already been considered, six further possible types of solution exist. These are

(iii) Ia, IIb, IIIb,

(iv) Ib, IIa, IIIb,

(v) Ib, IIb, IIIa,

(vi) Ia, IIa, IIIb,

(vii) Ia, IIb, IIIa

and (viii) Ib, IIa, IIIa.

Clearly, in each of the cases (iii), (iv), ..., (viii), when $p = 0$, $\lambda_1\lambda_2\lambda_3 = 0$ and when $p = 3f_3^2/4E$, $\lambda_1\lambda_2\lambda_3 > 0$. However, $\lambda_1\lambda_2\lambda_3$ need not necessarily increase monotonically as p increases from 0 to $3f_3^2/4E$.

Let us denote by λ_{1a} and λ_{1b}, the lower and upper values respectively of λ_1 found from equation (3·3), with analogous meanings for λ_{2a} and λ_{2b} and for λ_{3a} and λ_{3b}.

Then, noting that
$$\lambda_{1a} < \lambda_{2a} < \lambda_{3a} < \lambda_{3b} < \lambda_{2b} < \lambda_{1b},$$

we see that, for any positive value of p less than $3f_3^2/4E$,

$$\lambda_{1a}\lambda_{2b}\lambda_{3b} < \lambda_{1b}\lambda_{2a}\lambda_{3b} < \lambda_{1b}\lambda_{2b}\lambda_{3a} < \lambda_{1b}\lambda_{2b}\lambda_{3b},$$

and
$$\lambda_{1a}\lambda_{2a}\lambda_{3a} < \lambda_{1a}\lambda_{2a}\lambda_{3b} < \lambda_{1a}\lambda_{2b}\lambda_{3a} < \lambda_{1b}\lambda_{2a}\lambda_{3a}.$$

Also
$$\lambda_{1a}\lambda_{2b}\lambda_{3a} < \lambda_{1a}\lambda_{2b}\lambda_{3b} \quad \text{and} \quad \lambda_{1b}\lambda_{2a}\lambda_{3a} < \lambda_{1b}\lambda_{2a}\lambda_{3b}.$$

Thus, noting that, when $p = 0$,

$$\lambda_{1a}\lambda_{2a}\lambda_{3a} = \lambda_{1a}\lambda_{2a}\lambda_{3b} = \lambda_{1a}\lambda_{2b}\lambda_{3a} = \lambda_{1b}\lambda_{2a}\lambda_{3a} = \lambda_{1a}\lambda_{2b}\lambda_{3b} = \lambda_{1b}\lambda_{2a}\lambda_{3b} = \lambda_{1b}\lambda_{2b}\lambda_{3a} = 0,$$

we see that if a solution of the type (i) exists, i.e. $\lambda_{1a}\lambda_{2a}\lambda_{3a} = 1$, for some value of p, a solution of each of the other types, except (ii), must exist. Again, if a solution of the type (vi) exists, a solution of each of the types, (iii), (iv), (v), (vii) and (viii) must exist, and so on.

We thus see that multiple solutions can occur, and that the number of these depends on the values of f_1/E, f_2/E and f_3/E. Of course, before we can be sure that all these represent physically possible states of strain, it is necessary to investigate the stability of the various equilibrium conditions.

4. Conditions for the existence of the various types of solution

It has been seen in § 3 that condition (3·7) is necessary and sufficient for the existence of a solution of the type I*a*, II*a*, III*a* and condition (3·6) is necessary and sufficient for the existence of a solution of the type I*b*, II*b*, III*b*. In this section the conditions for the existence of the remaining six types of solution will be discussed. We shall discuss the case of the I*a*, II*b*, III*b* type of solution in detail and shall merely give conclusions in the other cases, since the method of treating them is analogous.

Let us consider

$$\Lambda = \lambda_{1a}\lambda_{2b}\lambda_{3b}. \tag{4·1}$$

The greatest value of p, which gives real values to λ_{1a}, λ_{2b} and λ_{3b} is $3f_3^2/4E$. As p increases from 0 to this value, Λ may not increase monotonically. Suppose p_3 is the value of p, in the range 0 to $3f_3^2/4E$, for which Λ has its greatest value. Then, provided the value of Λ, when $p = p_3$, is greater than unity, there will be at least one value of p, less than p_3, for which $\Lambda = 1$, yielding a solution of the type I*a*, II*b*, III*b*. There may, of course, also be some value of p, greater than p_3, for which $\Lambda = 1$. However, the condition

$$[\Lambda]_{p=p_3} > 1 \tag{4·2}$$

implies that there is at least one solution of the type I*a*, II*b*, III*b* and is a necessary condition for the existence of such a solution.

When $p = p_3$,

$$\left.\begin{aligned} 2E\lambda_{1a} &= 3[f_1 - (f_1^2 - \tfrac{4}{3}Ep_3)^{\frac{1}{2}}], \\ 2E\lambda_{2b} &= 3[f_2 + (f_2^2 - \tfrac{4}{3}Ep_3)^{\frac{1}{2}}] \\ \text{and} \quad 2E\lambda_{3b} &= 3[f_3 + (f_3^2 - \tfrac{4}{3}Ep_3)^{\frac{1}{2}}]. \end{aligned}\right\} \tag{4·3}$$

Thus, the condition (4·2) becomes

$$\left(\frac{3}{2E}\right)^3 [f_1 - (f_1^2 - \tfrac{4}{3}Ep_3)^{\frac{1}{2}}]\,[f_2 + (f_2^2 - \tfrac{4}{3}Ep_3)^{\frac{1}{2}}]\,[f_3 + (f_3^2 - \tfrac{4}{3}Ep_3)^{\frac{1}{2}}] > 1. \tag{4·4}$$

In a similar manner, the necessary and sufficient conditions for the existence of at least one solution of each of the remaining types can be found. They are tabulated in table 1, $p_4, p_5, \ldots, p_8$ being the values of p, in the range 0 to $3f_3^2/4E$ for which Λ has its greatest value, in the cases (iv), (v), ..., (viii) respectively.

TABLE 1

type	necessary and sufficient condition for solution
I*a*, II*a*, III*a*	$\left(\frac{3}{2E}\right)^3 f_3[f_2-(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}]>1$
I*b*, II*b*, III*b*	$\left(\frac{3}{2E}\right)^3 f_3[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1+(f_1^2-f_3^2)^{\frac{1}{2}}]<1$
I*a*, II*b*, III*b*	$\left(\frac{3}{2E}\right)^3 [f_1-(f_1^2-\frac{4}{3}Ep_3)^{\frac{1}{2}}]\,[f_2+(f_2^2-\frac{4}{3}Ep_3)^{\frac{1}{2}}]\,[f_3+(f_3^2-\frac{4}{3}Ep_3)^{\frac{1}{2}}]>1$
I*b*, II*a*, III*b*	$\left(\frac{3}{2E}\right)^3 [f_1+(f_1^2-\frac{4}{3}Ep_4)^{\frac{1}{2}}]\,[f_2-(f_2^2-\frac{4}{3}Ep_4)^{\frac{1}{2}}]\,[f_3+(f_3^2-\frac{4}{3}Ep_4)^{\frac{1}{2}}]>1$
I*b*, II*b*, III*a*	$\left(\frac{3}{2E}\right)^3 [f_1+(f_1^2-\frac{4}{3}Ep_5)^{\frac{1}{2}}]\,[f_2+(f_2^2-\frac{4}{3}Ep_5)^{\frac{1}{2}}]\,[f_3-(f_3^2-\frac{4}{3}Ep_5)^{\frac{1}{2}}]>1$
I*a*, II*a*, III*b*	$\left(\frac{3}{2E}\right)^3 [f_1-(f_1^2-\frac{4}{3}Ep_6)^{\frac{1}{2}}]\,[f_2-(f_2^2-\frac{4}{3}Ep_6)^{\frac{1}{2}}]\,[f_3+(f_3^2-\frac{4}{3}Ep_6)^{\frac{1}{2}}]>1$
I*a*, II*b*, III*a*	$\left(\frac{3}{2E}\right)^3 [f_1-(f_1^2-\frac{4}{3}Ep_7)^{\frac{1}{2}}]\,[f_2+(f_2^2-\frac{4}{3}Ep_7)^{\frac{1}{2}}]\,[f_3-(f_3^2-\frac{4}{3}Ep_7)^{\frac{1}{2}}]>1$
I*b*, II*a*, III*a*	$\left(\frac{3}{2E}\right)^3 [f_1+(f_1^2-\frac{4}{3}Ep_8)^{\frac{1}{2}}]\,[f_2-(f_2^2-\frac{4}{3}Ep_8)^{\frac{1}{2}}]\,[f_3-(f_3^2-\frac{4}{3}Ep_8)^{\frac{1}{2}}]>1$

The value of p_3 can, in theory, be found in the following manner. From (4·1) and (4·3),

$$\Lambda=\left(\frac{3}{2E}\right)^3[f_1-(f_1^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}]\,[f_2+(f_2^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}]\,[f_3+(f_3^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}]. \tag{4·5}$$

Λ has stationary values when $d\Lambda/dp=0$.

From (4·5),

$$\log\Lambda=3\log\frac{3}{2E}+\log[f_1-(f_1^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}]+\log[f_2+(f_2^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}]+\log[f_3+(f_3^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}].$$

Therefore

$$\frac{d\Lambda}{dp}=-\Lambda\left[\frac{1}{\lambda_{2b}(f_2^2-\frac{4}{3}Ep)^{\frac{1}{2}}}+\frac{1}{\lambda_{3b}(f_3^2-\frac{4}{3}Ep)^{\frac{1}{2}}}-\frac{1}{\lambda_{1a}(f_1^2-\frac{4}{3}Ep)^{\frac{1}{2}}}\right]. \tag{4·6}$$

The equation $d\Lambda/dp=0$ cannot, in general, be readily solved analytically for p. Consequently, we cannot make the necessary and sufficient condition (4·4), for the existence of a solution of the type I*a*, II*b*, III*b*, independent of p_3. However, a sufficient condition, which is less stringent than (4·4), can be readily obtained. This depends on the fact that if $\Lambda>1$ for $p=3f_3^2/4E$, a solution of the type I*a*, II*b*, III*b* exists. When $p=3f_3^2/4E$,

$$\Lambda=\left(\frac{3}{2E}\right)^3[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}]\,[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]f_3,$$

and the sufficient condition for the existence of a solution of the type I*a*, II*b*, III*b* is

$$\left(\frac{3}{2E}\right)^3[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}]\,[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]f_3>1.$$

In a similar manner sufficient conditions for the existence of the various other types of solution can be obtained and are tabulated in table 2.

If $dA/dp = 0$ has no solution for $0<p<3f_3^2/4E$ then A increases monotonically as p increases from 0 to $3f_3^2/4E$. The greatest value of A occurs when $p = 3f_3^2/4E$ and the sufficient condition in table 2 becomes, as well, a necessary condition.

TABLE 2

type	sufficient condition for solution
Ia, IIa, IIIa	$\left(\frac{3}{2E}\right)^3 f_3[f_2-(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}]>1$
Ib, IIb, IIIb	$\left(\frac{3}{2E}\right)^3 f_3[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]\,[f_1+(f_1^2-f_3^2)^{\frac{1}{2}}]<1$
Ia, IIb, IIIb	$\left(\frac{3}{2E}\right)^3 f_3[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}]\,[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]>1$
Ib, IIa, IIIb	$\left(\frac{3}{2E}\right)^3 f_3[f_1+(f_1^2-f_3^2)^{\frac{1}{2}}]\,[f_2-(f_2^2-f_3^2)^{\frac{1}{2}}]>1$
Ib, IIb, IIIa	$\left(\frac{3}{2E}\right)^3 f_3[f_1+(f_1^2-f_3^2)^{\frac{1}{2}}]\,[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]>1$
Ia, IIa, IIIb	$\left(\frac{3}{2E}\right)^3 f_3[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}]\,[f_2-(f_2^2-f_3^2)^{\frac{1}{2}}]>1$
Ia, IIb, IIIa	$\left(\frac{3}{2E}\right)^3 f_3[f_1-(f_1^2-f_3^2)^{\frac{1}{2}}]\,[f_2+(f_2^2-f_3^2)^{\frac{1}{2}}]>1$
Ib, IIa, IIIa	$\left(\frac{3}{2E}\right)^3 f_3[f_1+(f_1^2-f_3^2)^{\frac{1}{2}}]\,[f_2-(f_2^2-f_3^2)^{\frac{1}{2}}]>1$

Returning to equation (4·6), for the Ia, IIb, IIIb case, we see that as $p \to 3f_3^2/4E$, from below, $dA/dp \to -\infty$. Since, when $p = 0$, $dA/dp > 0$, and since A is a continuous function of p in the range $0<p<3f_3^2/4E$, A must have at least one maximum for $0<p<3f_3^2/4E$ and if there is more than one stationary value for p, there is an odd number. For the Ib, IIa, IIIb and Ia, IIa, IIIb cases, a similar argument applies.

However, in the Ib, IIb, IIIa case,

$$\frac{dA}{dp} = -A\left[\frac{1}{\lambda_{1b}(f_1^2-\frac{4}{3}Ep)^{\frac{1}{2}}}+\frac{1}{\lambda_{2b}(f_2^2-\frac{4}{3}Ep)^{\frac{1}{2}}}-\frac{1}{\lambda_{3a}(f_3^2-\frac{4}{3}Ep)^{\frac{1}{2}}}\right]. \tag{4·7}$$

As $p \to 3f_3^2/4E$, from below, $dA/dp \to \infty$ and consequently it cannot be definitely stated that there is at least one maximum for $0<p<3f_3^2/4E$. A similar argument applies for the Ib, IIa, IIIa and Ia, IIb, IIIa cases.

5. THE STABILITY OF EQUILIBRIUM (PURE, HOMOGENEOUS STRAIN UNDER SPECIFIED FORCES)

If a cube of the incompressible, neo-Hookean material, which has unit edge in the unstrained state, is strained to dimensions $\lambda_1 \times \lambda_2 \times \lambda_3$ under the action of three forces f_1, f_2 and f_3, the work done by these forces is

$$f_1(\lambda_1-1)+f_2(\lambda_2-1)+f_3(\lambda_3-1). \tag{5·1}$$

The energy stored elastically, in the material, is, from equation (I, 9·3),

$$\tfrac{1}{6}E(\lambda_1^2+\lambda_2^2+\lambda_3^2-3). \tag{5·2}$$

Writing $$\Phi = \tfrac{1}{6}E(\lambda_1^2+\lambda_2^2+\lambda_3^2-3)-[f_1(\lambda_1-1)+f_2(\lambda_2-1)+f_3(\lambda_3-1)], \tag{5·3}$$

the equilibrium conditions for pure, homogeneous deformation of the body are given by

$$\delta\Phi = 0, \tag{5·4}$$

for all possible small variations of λ_1, λ_2 and λ_3.

Since $$\lambda_1\lambda_2\lambda_3 = 1, \tag{5·5}$$

the allowable variations $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ of λ_1, λ_2, λ_3 respectively are restricted by the relation

$$\lambda_2\lambda_3\delta\lambda_1+\lambda_3\lambda_1\delta\lambda_2+\lambda_1\lambda_2\delta\lambda_3 = 0,$$

i.e. by $$\frac{\delta\lambda_1}{\lambda_1}+\frac{\delta\lambda_2}{\lambda_2}+\frac{\delta\lambda_3}{\lambda_3} = 0. \tag{5·6}$$

The equilibrium condition is stable if

$$\delta^2\Phi > 0, \tag{5·7}$$

for all allowable, small variations of $\delta\lambda_1$, $\delta\lambda_2$ and $\delta\lambda_3$ about the equilibrium position. In order to avoid introducing the restricting condition (5·6), we may rewrite (5·3) as

$$\Phi = \tfrac{1}{6}E\left(\lambda_1^2+\lambda_2^2+\frac{1}{\lambda_1^2\lambda_2^2}\right)-\left[f_1(\lambda_1-1)+f_2(\lambda_2-1)+f_3\left(\frac{1}{\lambda_1\lambda_2}-1\right)\right]. \tag{5·8}$$

Then, all small variations of λ_1, λ_2 are allowable. Now

$$\delta^2\Phi = \frac{\partial^2\Phi}{\partial\lambda_1^2}(\delta\lambda_1)^2+2\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}(\delta\lambda_1\delta\lambda_2)+\frac{\partial^2\Phi}{\partial\lambda_2^2}(\delta\lambda_2)^2. \tag{5·9}$$

From (5·8), $$\frac{\partial^2\Phi}{\partial\lambda_1^2} = \tfrac{1}{3}E\left(1+\frac{3}{\lambda_1^4\lambda_2^2}\right)-2f_3\frac{1}{\lambda_1^3\lambda_2}$$

$$= \tfrac{1}{3}E\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)-2f_3\frac{\lambda_3}{\lambda_1^2}, \tag{5·10}$$

$$\frac{\partial^2\Phi}{\partial\lambda_2^2} = \tfrac{1}{3}E\left(1+\frac{3}{\lambda_1^2\lambda_2^4}\right)-2f_3\frac{1}{\lambda_1\lambda_2^3}$$

$$= \tfrac{1}{3}E\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)-2f_3\frac{\lambda_3}{\lambda_2^2}, \tag{5·11}$$

$$\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2} = \tfrac{2}{3}E\frac{1}{\lambda_1^3\lambda_2^3}-f_3\frac{1}{\lambda_1^2\lambda_2^2}$$

$$= (\tfrac{2}{3}E\lambda_3-f_3)\lambda_3^2. \tag{5·12}$$

The necessary and sufficient condition for $\delta^2\Phi$, as given in equation (5·9), to be greater than zero, for all possible small values of $\delta\lambda_1$ and $\delta\lambda_2$, is

$$\frac{\partial^2\Phi}{\partial\lambda_1^2}\frac{\partial^2\Phi}{\partial\lambda_2^2} > \left(\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}\right)^2, \tag{5·13}$$

$$\frac{\partial^2\Phi}{\partial\lambda_1^2} > 0 \quad\text{and}\quad \frac{\partial^2\Phi}{\partial\lambda_2^2} > 0. \tag{5·14}$$

Either of the conditions (5·14) is, of course, implied by the other, together with condition (5·13).

Introducing the expressions for $\partial^2\Phi/\partial\lambda_1^2$, $\partial^2\Phi/\partial\lambda_2^2$ and $\partial^2\Phi/\partial\lambda_1\,\partial\lambda_2$, given in equations (5·10), (5·11) and (5·12), the condition (5·13) becomes

$$\left[\tfrac{1}{3}E\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)-2f_3\frac{\lambda_3}{\lambda_1^2}\right]\left[\tfrac{1}{3}E\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)-2f_3\frac{\lambda_3}{\lambda_2^2}\right]>(\tfrac{2}{3}E\lambda_3-f_3)^2\,\lambda_3^4, \tag{5·15}$$

and the conditions (5·14) become

$$\tfrac{1}{3}E\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)>2f_3\frac{\lambda_3}{\lambda_1^2} \quad\text{and}\quad \tfrac{1}{3}E\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)>2f_3\frac{\lambda_3}{\lambda_2^2}. \tag{5·16}$$

Bearing in mind that $$f_3\lambda_3=\tfrac{1}{3}E\lambda_3^2+p,$$

condition (5·15) becomes

$$\tfrac{1}{9}E^2\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}\right)-\tfrac{2}{3}Ep(\lambda_1^2+\lambda_2^2+\lambda_3^2)+3p^2>0, \tag{5·17}$$

or $$(\tfrac{1}{3}E\lambda_1^2-p)\,(\tfrac{1}{3}E\lambda_2^2-p)+(\tfrac{1}{3}E\lambda_2^2-p)\,(\tfrac{1}{3}E\lambda_3^2-p)+(\tfrac{1}{3}E\lambda_3^2-p)\,(\tfrac{1}{3}E\lambda_1^2-p)>0, \tag{5·18}$$

and conditions (5·16) become

$$\tfrac{1}{3}E(\lambda_1^2+\lambda_3^2)>2p \tag{5·19}$$

and $$\tfrac{1}{3}E(\lambda_2^2+\lambda_3^2)>2p. \tag{5·20}$$

We notice that if $$p<\tfrac{1}{3}E\lambda_1^2,\quad p<\tfrac{1}{3}E\lambda_2^2\quad\text{and}\quad p<\tfrac{1}{3}E\lambda_3^2, \tag{5·21}$$

then all three of the necessary and sufficient conditions for stability (5·18), (5·19) and (5·20) are satisfied and the corresponding equilibrium state is stable. Condition (5·21) is, of course, automatically satisfied if p is negative, as in the case considered in § 2, where the equilibrium state is uniquely determined.

In deducing the stability conditions, we could have expressed $\delta^2\Phi$ in terms of λ_2, λ_3, $\delta\lambda_2$ and $\delta\lambda_3$ instead of λ_1, λ_2, $\delta\lambda_1$ and $\delta\lambda_2$. We should then have obtained as our stability conditions

$$\tfrac{1}{3}E(\lambda_2^2+\lambda_1^2)>2p \quad\text{and}\quad \tfrac{1}{3}E(\lambda_3^2+\lambda_1^2)>2p,$$

together with condition (5·18).

It therefore appears that we can add to conditions (5·18) and (5·19) the condition

$$\tfrac{1}{3}E(\lambda_1^2+\lambda_2^2)>2p \tag{5·22}$$

as a necessary condition.*

* *Footnote* (*added* 8 *May* 1946).

It has been pointed out to me by Dr D. N. de G. Allen that if the condition (5·18) and any one of the three conditions (5·19), (5·20) and (5·22) are satisfied, then it follows that the remaining two of these conditions are automatically satisfied.

Thus, suppose (5·18) and (5·19) are satisfied. Adding the positive quantity $(\tfrac{1}{3}E\lambda_3^2-p)^2$ to the left-hand side of (5·18), it follows that

$$[\tfrac{1}{3}E(\lambda_2^2+\lambda_3^2)-2p]\,[\tfrac{1}{3}E(\lambda_1^2+\lambda_3^2)-2p]>0.$$

From this and (5·19) it follows that

$$\tfrac{1}{3}E(\lambda_2^2+\lambda_3^2)-2p>0.$$

In a similar manner, it can be shown that (5·22) is satisfied.

6. The stability of equilibrium (pure, homogeneous strain under specified stresses)

If the normal stresses t_{xx}, t_{yy} and t_{zz} acting on the cube of incompressible, neo-Hookean material are specified, then it has been seen, from § 6 of Part I, that the equilibrium state is uniquely determined. It can readily be shown, in a manner similar to that of § 5, that this equilibrium state is necessarily stable.

In this case the forces acting on opposite pairs of faces of the strained cube are $t_{xx}\lambda_2\lambda_3$, $t_{yy}\lambda_3\lambda_1$ and $t_{zz}\lambda_1\lambda_2$. The work done by these forces in a small, virtual change of dimensions of the cuboid from $\lambda_1\times\lambda_2\times\lambda_3$ to $(\lambda_1+\delta\lambda_1)\times(\lambda_2+\delta\lambda_2)\times(\lambda_3+\delta\lambda_3)$ is

$$t_{xx}\lambda_2\lambda_3\delta\lambda_1+t_{yy}\lambda_3\lambda_1\delta\lambda_2+t_{zz}\lambda_1\lambda_2\delta\lambda_3=t_{xx}\frac{\delta\lambda_1}{\lambda_1}+t_{yy}\frac{\delta\lambda_2}{\lambda_2}+t_{zz}\frac{\delta\lambda_3}{\lambda_3},$$

in view of the relationship (5·5), for an incompressible material.

Now
$$\delta\Phi=\delta[\tfrac{1}{6}E(\lambda_1^2+\lambda_2^2+\lambda_3^2-3)]-\left[t_{xx}\frac{\delta\lambda_1}{\lambda_1}+t_{yy}\frac{\delta\lambda_2}{\lambda_2}+t_{zz}\frac{\delta\lambda_3}{\lambda_3}\right]$$
$$=\tfrac{1}{3}E(\lambda_1\delta\lambda_1+\lambda_2\delta\lambda_2+\lambda_3\delta\lambda_3)-\left(t_{xx}\frac{\delta\lambda_1}{\lambda_1}+t_{yy}\frac{\delta\lambda_2}{\lambda_2}+t_{zz}\frac{\delta\lambda_3}{\lambda_3}\right).$$

Making use of the relationships (5·5) and (5·6), to eliminate λ_3 and $\delta\lambda_3$, we have

$$\delta\Phi=\tfrac{1}{3}E\left[\left(\lambda_1-\frac{1}{\lambda_1^3\lambda_2^2}\right)\delta\lambda_1+\left(\lambda_2-\frac{1}{\lambda_2^3\lambda_1^2}\right)\delta\lambda_2\right]-\left[(t_{xx}-t_{zz})\frac{\delta\lambda_1}{\lambda_1}+(t_{yy}-t_{zz})\frac{\delta\lambda_2}{\lambda_2}\right].$$

Whence
$$\delta^2\Phi=\tfrac{1}{3}E\left[\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)(\delta\lambda_1)^2+4\lambda_3^3\,\delta\lambda_1\,\delta\lambda_2+\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)(\delta\lambda_2)^2\right]$$
$$+\left[(t_{xx}-t_{zz})\frac{(\delta\lambda_1)^2}{\lambda_1^2}+(t_{yy}-t_{zz})\frac{(\delta\lambda_2)^2}{\lambda_2^2}\right]. \quad (6\cdot1)$$

For the equilibrium state, we have, from equations (I, 3·5)

$$t_{xx}-t_{zz}=\tfrac{1}{3}E(\lambda_1^2-\lambda_3^2)\quad\text{and}\quad t_{yy}-t_{zz}=\tfrac{1}{3}E(\lambda_2^2-\lambda_3^2).$$

Introducing these results into (6·1),

$$\delta^2\Phi=\tfrac{2}{3}E\left[\left(1+\frac{\lambda_3^2}{\lambda_1^2}\right)(\delta\lambda_1)^2+2\lambda_3^3\delta\lambda_1\,\delta\lambda_2+\left(1+\frac{\lambda_3^2}{\lambda_2^2}\right)(\delta\lambda_2)^2\right]$$
$$=\tfrac{2}{3}E\left[(\delta\lambda_1)^2+(\delta\lambda_2)^2+\lambda_3^2\left(\frac{\delta\lambda_1}{\lambda_1}+\frac{\delta\lambda_2}{\lambda_2}\right)^2\right].$$

Thus $\delta^2\Phi>0$, and the equilibrium state is stable.

7. The unstable equilibrium states

It can readily be shown that the states of equilibrium given by solutions of the types I*a*, II*a*, III*a*, I*a*, II*a*, III*b*, I*a*, II*b*, III*a* and I*b*, II*a*, III*a* are unstable.

The necessary and sufficient conditions for stable equilibrium have been shown to be (5·18), (5·19), (5·20) and (5·22).

If one or more of these conditions is not satisfied, for some state of equilibrium, then that state is unstable.

For a solution of the type I*a*, II*a*, III*a*,

$$2E\lambda_i = 3[f_i - (f_i^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}] \quad (i = 1, 2, 3).$$

Now, since

$$f_1 - (f_1^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}} < f_1 + (f_1^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}},$$

$$\lambda_1^2 < \left(\frac{3}{2E}\right)^2 [f_1 - (f_1^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}][f_1 + (f_1^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}]$$

i.e.

$$\tfrac{1}{3}E\lambda_1^2 - p < 0.$$

Similarly, it may be shown that

$$\tfrac{1}{3}E\lambda_2^2 - p < 0 \quad \text{and} \quad \tfrac{1}{3}E\lambda_3^2 - p < 0.$$

Thus,

$$\tfrac{1}{3}E(\lambda_1^2 + \lambda_3^2) < 2p, \quad \tfrac{1}{3}E(\lambda_2^2 + \lambda_3^2) < 2p \quad \text{and} \quad \tfrac{1}{3}E(\lambda_1^2 + \lambda_2^2) < 2p,$$

in disagreement with the conditions for stability (5·19), (5·20) and (5·22). Thus the equilibrium condition represented by a solution of the type I*a*, II*a*, III*a* is necessarily unstable.

In a similar manner, it can be shown that the solutions of the types I*a*, II*a*, III*b*, I*a*, II*b*, III*a* and I*b*, II*a*, III*a* contradict the stability conditions (5·22), (5·19) and (5·20) respectively and therefore correspond to states of unstable equilibrium.

8. The stable equilibrium states

(*a*) *Consideration of the* I*b*, II*b*, III*b* *equilibrium state*

In this state,

$$2E\lambda_i = 3[f_i + (f_i^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}] \quad (i = 1, 2, 3).$$

Now,

$$\tfrac{1}{3}E\lambda_i^2 - p > 0 \quad (i = 1, 2, 3).$$

Thus, conditions (5·21) are satisfied for the I*b*, II*b*, III*b* state of equilibrium, which is therefore always stable if it exists.

(*b*) *Consideration of the* I*b*, II*b*, III*a* *state*

For this state

$$2E\lambda_1 = 3[f_1 + (f_1^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}], \quad 2E\lambda_2 = 3[f_2 + (f_2^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}] \quad \text{and} \quad 2E\lambda_3 = 3[f_3 - (f_3^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}]. \tag{8·1}$$

We can readily see that

$$\tfrac{1}{3}E\lambda_1^2 - p > 0, \quad \tfrac{1}{3}E\lambda_2^2 - p > 0 \quad \text{and} \quad \tfrac{1}{3}E\lambda_3^2 - p < 0. \tag{8·2}$$

Condition (5·22) is clearly satisfied.

$$\tfrac{1}{3}E(\lambda_1^2 + \lambda_3^2) - 2p = \lambda_1(f_1^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}} - \lambda_3(f_3^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}} > 0,$$

since $\lambda_1 > \lambda_3$ and $f_1 > f_3$. Thus condition (5·19) is satisfied.

Similarly, since $\lambda_2 > \lambda_3$ and $f_2 > f_3$, condition (5·20) is satisfied.

In view of the relations (8·2), the stability condition (5·18) may be written, for the I*b*, II*b*, III*a* equilibrium state, thus:

$$\frac{1}{p - \frac{1}{3}E\lambda_3^2} > \frac{1}{\frac{1}{3}E\lambda_1^2 - p} + \frac{1}{\frac{1}{3}E\lambda_2^2 - p}. \tag{8·3}$$

Now, equations (8·1) give

$$\tfrac{1}{3}E\lambda_1^2-p=\lambda_1(f_1^2-\tfrac{4}{3}Ep)^{\frac{1}{2}},\quad \tfrac{1}{3}E\lambda_2^2-p=\lambda_2(f_2^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}\quad\text{and}\quad \tfrac{1}{3}E\lambda_3^2-p=-\lambda_3(f_3^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}. \tag{8·4}$$

So, condition (8·3) becomes

$$\frac{1}{\lambda_3(f_3^2-\frac{4}{3}Ep)^{\frac{1}{2}}}>\frac{1}{\lambda_1(f_1^2-\frac{4}{3}Ep)^{\frac{1}{2}}}+\frac{1}{\lambda_2(f_2^2-\frac{4}{3}Ep)^{\frac{1}{2}}}. \tag{8·5}$$

If $\Lambda=\lambda_1\lambda_2\lambda_3$, we can readily see that

$$\left[\frac{d\Lambda}{dp}\right]_{\Lambda=1}=\left[\frac{1}{\lambda_3(f_3^2-\frac{4}{3}Ep)^{\frac{1}{2}}}-\frac{1}{\lambda_2(f_2^2-\frac{4}{3}Ep)^{\frac{1}{2}}}-\frac{1}{\lambda_1(f_1^2-\frac{4}{3}Ep)^{\frac{1}{2}}}\right],$$

for this state.

Consequently, if $d\Lambda/dp>0$, for the equilibrium state considered, condition (5·18) is satisfied and the state is stable. On the other hand if $d\Lambda/dp<0$, for the equilibrium state considered, condition (5·18) is not satisfied and the state is unstable.

(c) *Consideration of the* I *b*, II *a*, III *b state*

For this state

$$2E\lambda_1=3[f_1+(f_1^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}],\quad 2E\lambda_2=3[f_2-(f_2^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}]\quad\text{and}\quad 2E\lambda_3=3[f_3+(f_3^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}].$$

We can readily see that condition (5·19) is satisfied, since

$$\tfrac{1}{3}E\lambda_1^2>p\quad\text{and}\quad \tfrac{1}{3}E\lambda_3^2>p.$$

Also, condition (5·22) becomes

$$\lambda_1(f_1^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}-\lambda_2(f_2^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}>0.$$

Since $\lambda_2<\lambda_1$ and $f_2<f_1$, this condition is also satisfied.

However, condition (5·20) becomes

$$\lambda_3(f_3^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}-\lambda_2(f_2^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}>0. \tag{8·6}$$

This condition may or may not be satisfied, depending on the values of f_1, f_2 and f_3.

Condition (5·18) becomes

$$\frac{1}{\lambda_2(f_2^2-\frac{4}{3}Ep)^{\frac{1}{2}}}>\frac{1}{\lambda_3(f_3^2-\frac{4}{3}Ep)^{\frac{1}{2}}}+\frac{1}{\lambda_1(f_1^2-\frac{4}{3}Ep)^{\frac{1}{2}}}. \tag{8·7}$$

If this condition is satisfied, then condition (8·6) and hence (5·20) is automatically satisfied.

As in the case of the I *b*, II *b*, III *a* equilibrium state, condition (8·7) is or is not satisfied, i.e. the equilibrium state is stable or unstable, accordingly as $d\Lambda/dp$ is positive or negative for the state considered.

(d) *Consideration of the* I *a*, II *b*, III *b state*

For this state

$$2E\lambda_1=3[f_1-(f_1^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}],\quad 2E\lambda_2=3[f_2+(f_2^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}]\quad\text{and}\quad 2E\lambda_3=3[f_3+(f_3^2-\tfrac{4}{3}Ep)^{\frac{1}{2}}].$$

As in the previous case, it can be shown that condition (5·20) is satisfied and that if condition (5·18) is satisfied, conditions (5·19) and (5·22) are also satisfied. Condition (5·18) becomes, in this case,

$$\frac{1}{\lambda_1(f_1^2-\frac{4}{3}Ep)^{\frac{1}{2}}}>\frac{1}{\lambda_2(f_2^2-\frac{4}{3}Ep)^{\frac{1}{2}}}+\frac{1}{\lambda_3(f_3^2-\frac{4}{3}Ep)^{\frac{1}{2}}}. \tag{8·8}$$

Again, as in the cases of the Ib, IIb, IIIa and Ib, IIa, IIIb states, condition (8·8) is or is not satisfied, i.e. the equilibrium is stable or unstable accordingly as $d\Lambda/dp$ is positive or negative for the state.

Summarizing the results obtained so far on the stability of the various types of equilibrium state, we have

(i) states of the types Ia, IIa, IIIa, Ia, IIa, IIIb, Ia, IIb, IIIa and Ib, IIa, IIIa are always unstable;

(ii) states of the types Ib, IIb, IIIb are always stable; and

(iii) states of the types Ia, IIb, IIIb, Ib, IIa, IIIb and Ib, IIb, IIIa are stable or unstable accordingly as $d(\lambda_1\lambda_2\lambda_3)/dp$ is positive or negative for the state.

9. Case of three equal forces

Suppose that the cube of material is subjected to three equal, positive forces each of magnitude f, so that

$$f_1 = f_2 = f_3 = f.$$

We shall assume that $f > \frac{1}{3}E$, so that condition (2·13) is satisfied and p is positive.

An equilibrium state of the type Ib, IIb, IIIb will exist if condition (3·6), which becomes

$$f < \tfrac{2}{3}E,$$

is satisfied. If such a state exists, then

$$\lambda_1 = \lambda_2 = \lambda_3 = 1,$$

for that state and it is stable (§ 8 (a)).

The necessary and sufficient condition for the existence of equilibrium states of the types Ia, IIb, IIIb, Ib, IIa, IIIb and Ib, IIb, IIIa, is (from table 1, § 4).

Thus,
$$\Lambda = \left(\frac{3}{2E}\right)^2 2p[f + (f^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}] > 1, \tag{9·1}$$

for some value of p between 0 and $3f^2/4E$.

$$\frac{d\Lambda}{dp} = \Lambda\left[\frac{1}{p} - \frac{\frac{2}{3}E}{\{f + (f^2 - \frac{4}{3}Ep)^{\frac{1}{2}}\}(f^2 - \frac{4}{3}Ep)^{\frac{1}{2}}}\right].$$

The maximum value of Λ is given by

$$\frac{d\Lambda}{dp} = 0,$$

which yields $p = 2f^2/3E$. We note that this value of p is less than $3f^2/4E$, the maximum value of p for which real values of λ_1, λ_2 and λ_3 are obtained.

Introducing $p = 2f^2/3E$ into condition (9·1), we obtain

$$f > (\tfrac{1}{4})^{\frac{1}{3}} E, \tag{9·2}$$

as the condition for the existence of equilibrium states of the types Ia, IIb, IIIb, Ib, IIa, IIIb and Ib, IIb, IIIa.

If this condition is satisfied, $d\Lambda/dp$ will be positive for $0 < p < 2f^2/3E$ and negative for $2f^2/3E < p < 3f^2/4E$. Thus, there will be one and only one value of p for which $\Lambda = 1$ and

$d\Lambda/dp$ is positive and the corresponding equilibrium state is stable. There will also be a value of p for which $\Lambda = 1$ and $d\Lambda/dp$ is negative, so that the corresponding equilibrium state is unstable, if $\Lambda < 1$ for $p = 3f^2/4E$.

When $$p = 3f^2/4E, \quad \Lambda = 27f^3/8E^3.$$

So $\Lambda < 1$ when $p = 3f^2/4E$, if $$f < \tfrac{2}{3}E. \tag{9·3}$$

There is only a small range of values of f where conditions (9·2) and (9·3) are consistent. In this range conditions (9·1) and (3·6) are satisfied, so that a single, stable equilibrium state of each of the four types Ib, IIb, IIIb, Ia, IIb, IIIb, Ib, IIa, IIIb and Ib, IIb, IIIa exists. Outside this range either one stable equilibrium state of the type Ib, IIb, IIIb (giving $\lambda_1 = \lambda_2 = \lambda_3$) exists or one stable equilibrium state of each of the three types Ia, IIb, IIIb, Ib, IIa, IIIb and Ib, IIb, IIIa exists. Any equilibrium states of the type Ia, IIa, IIIb, Ia, IIb, IIIa or Ib, IIa, IIIa, which exist, are of necessity unstable (§ 7).

10. Effect of order of application of forces

Let us suppose that the forces f_1 and f_2 are applied first and the body is allowed to reach its equilibrium state under the action of these forces. This equilibrium state is uniquely determined by f_1 and f_2 (§ 2). The dimensions λ_1', λ_2', λ_3' of the body in this state are given by

$$f_1\lambda_1' = \tfrac{1}{3}E\lambda_1'^2 + p', \quad f_2\lambda_2' = \tfrac{1}{3}E\lambda_2'^2 + p', \quad 0 = \tfrac{1}{3}E\lambda_3'^2 + p' \quad \text{and} \quad \lambda_1'\lambda_2'\lambda_3' = 1. \tag{10·1}$$

We have, from the first three of these equations,

$$\left.\begin{aligned} 2E\lambda_1' &= 3[f_1 + (f_1^2 - \tfrac{4}{3}Ep')^{\frac{1}{2}}],\\ 2E\lambda_2' &= 3[f_2 + (f_2^2 - \tfrac{4}{3}Ep')^{\frac{1}{2}}]\\ \text{and} \qquad 2E\lambda_3' &= 3(-\tfrac{4}{3}Ep')^{\frac{1}{2}}. \end{aligned}\right\} \tag{10·2}$$

Only the positive roots need be considered in solving the first three of equations (10·1) for λ_1', λ_2' and λ_3', for only these roots will lead to positive values of λ_1', λ_2' and λ_3', as explained in § 2. Furthermore, it is seen that p' must be negative. Otherwise λ_3' is imaginary.

Suppose that under the action of the three forces f_1, f_2, f_3 there exists a stable equilibrium of the type Ib, IIb, IIIb and a single equilibrium state of each of the types Ia, IIb, IIIb, Ib, IIa, IIIb and Ib, IIb, IIIa. Let us suppose that when the body is in the state represented by equations (10·1) the force on the faces, normal to the direction of the edge of length λ_3', is increased to its final value f_3 in such a manner that the equilibrium state is reached quasi-statically. The question arises—which of the four possible equilibrium states will be assumed by the body?

For the Ib, IIb, IIIa state of equilibrium

$$\left.\begin{aligned} 2E\lambda_1 &= 3[f_1 + (f_1^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}],\\ 2E\lambda_2 &= 3[f_2 + (f_2^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}]\\ \text{and} \qquad 2E\lambda_3 &= 3[f_3 - (f_3^2 - \tfrac{4}{3}Ep)^{\frac{1}{2}}]. \end{aligned}\right\} \tag{10·3}$$

Here, p is positive.

Therefore, since p' is negative,

$$\lambda_1 < \lambda_1', \quad \lambda_2 < \lambda_2' \quad \text{and} \quad \lambda_3 > \lambda_3'. \tag{10·4}$$

It can readily be seen that the relations (10·4) are also valid for equilibrium states of the types I*b*, II*a*, III*b*, I*a*, II*b*, III*a* and I*b*, II*b*, III*b*.

Now, λ_3 has a lower value in the state of the type I*b*, II*b*, III*a* than in any of the other three states. Therefore, the state of the type I*b*, II*b*, III*a* can be reached from the state in which the dimensions of the cuboid are λ_1', λ_2', λ_3', without passing through any other state of equilibrium, under the action of the forces f_1, f_2, f_3.

In a similar manner it can be shown that if the forces f_1 and f_3 are applied first, the state of the type I*b*, II*a*, III*b* can be reached without passing through any other equilibrium state. Again, if the forces f_2 and f_3 are applied first, the state of the type I*a*, II*b*, III*b* can be reached without passing through any other equilibrium state.

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